Workshop on New Approaches to the Phase Problem for Non-Periodic Objects
May 17 – 19, 2001

Phase retrieval and zero information

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Noushin, A.J., M. A. Fiddy and J. Graham-Eagle, "Some new findings on the zeros of bandlimited functions", J. Opt. Soc. Amer. A.16, pp 1857-1863, 1999

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Chen, P-T, M. A. Fiddy, C-W. Liao and D. A. Pommet, "Blind deconvolution and phase retrieval using point zeros", J. Opt. Soc. of Amer. A, 13, pp1524-1531, 1996.

PHASE RETRIEVAL

$$\text{F.T.}\{f\} = F = |F|e^{i\phi}$$

COMPACT SUPPORT

ENTIRE FUNCTION OF EXP. TYPE

$$I = FF^*$$

HISTORY:

1-D	ISOLATED ZEROS OF $F(x+iy)$
$\geq$ 2-D	ZEROS NOT ISOLATED
REAL DATA	HOW RELEVANT IS MODEL?

One-dimensional Phase Retrieval Problem

The field can be represented by a one-dimensional entire function $F(\zeta)$.

The Cauchy-Riemann equations must be satisfied :

$$\frac{\partial \mathrm{Re}[F(\zeta)]}{\partial u} = \frac{\partial \mathrm{Im}[F(\zeta)]}{\partial q},$$

$$\frac{\partial \mathrm{Re}[F(\zeta)]}{\partial q} = -\frac{\partial \mathrm{Im}[F(\zeta)]}{\partial u}$$

This gives strict relationships between the behavior of the real and imaginary parts of an analytic function.

$F(\zeta)$ can be represented by its zero points in product form. This product is known as the Hadamard product,

$$F(\zeta) = A\prod_{j=1}^{\infty}(1-\zeta/\zeta_j)$$

where A is a normalization factor and ζ_j is the jth zero location of the function $F(\zeta)$.

Fourier transform of a one dimensional signal or image of compact support is an entire function of exponential type

- can be represented in terms of their real and complex point zero locations by means of an infinite product of factors (the Hadamard product)

- complex zeros of the spectrum and its complex conjugate are located symmetrically about the real axis

- if N complex zeros located, then 2^{N-1} distinct complex functions consistent with the measured intensity

ONE DIMENSION

$$\log F(z) = \log |F(z)| + i\,\phi(z)$$

if $F(z) = |F| e^{i\phi}$

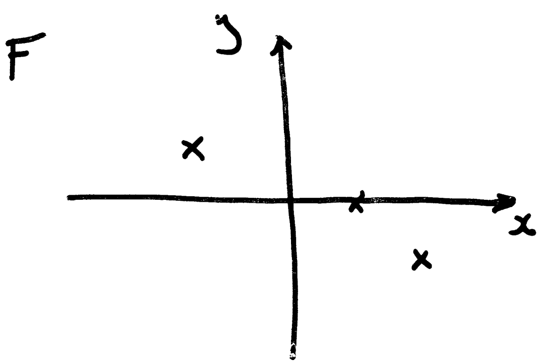
LHT $\quad \phi(x) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\log |F(x')|}{x'-x}\, dx'$

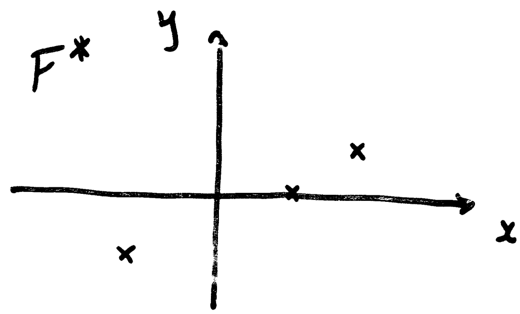
Contour integral $\rightarrow$ zeros in upper half plane
a problem

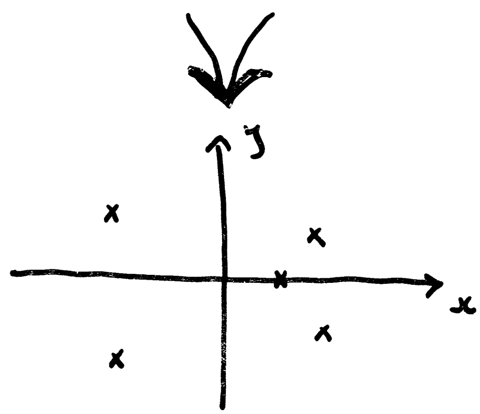
Hadamard Product $\quad F(z) = \prod_{j=1}^{\infty} \left(1 - z/z_j\right)$

Zero flipping

$$|F|^2 \equiv F F^*$$







Two-dimensional Phase Retrieval Problem

A 2-D bandlimited function can be represented by its zero structure. (Osgood product)

$$F(z_1,z_2)=\prod_{j=1}^{N}[F_j(z_1,z_2)e^{\gamma_j(z_1,z_2)}]^{L_j}$$

Where e^{γ_j} are referred to as convergence factors, γ_j is a polynomial and L_j is an integer.

If the spectrum $F(u,v)$ is a reducible function, for example $F(u,v)=F_1(u,v)\cdot F_2(u,v)$,

then

$$G_1(u,v)=F_1(u,v)\cdot F_2(u,v)$$
$$G_2(u,v)=F_1(u,v)\cdot F_2^*(u,v)$$
$$G_3(u,v)=F_1^*(u,v)\cdot F_2(u,v)$$
$$G_4(u,v)=F_1^*(u,v)\cdot F_2^*(u,v)$$

all have the same Fourier intensity, this cause ambiguities.

Since the 2-D bandlimited functions are general irreducible, reducible functions form a set of measure zero in the space of entire functions, we are expecting the phase retrieval problem have a unique solution.

Fourier transform in more than one dimension:

generally not factorizable into an infinite product of terms and is irreducible

irreducible factor encodes a zero structure which weaves itself through the associated complex space, of which the real plane is the only surface on which data are measured.

<u>Methods for phase retrieval</u>

Direct.....slow
Iterative...convergence problems

WHEN IS PHASE ENCODED IN $|F|$?

1-D GENERALLY NO UNIQUE SOL^N

2-D UNIQUE BECAUSE IRREDUCIBLE?

$F F^* +$ NOISE ?

PROBLEM OF LIMITED SAMPLED NOISY DATA .....

ITERATIVE METHODS

MINIMUM PHASE CONDITION $\left\{ \begin{array}{l} \text{ROUCHE'S} \\ \text{THEOREM} \end{array} \right.$

HOLOGRAPHY

LOCATE ZEROS ?

ONLY REAL ZEROS / ZERO CROSSINGS

When can phase be directly recovered from intensity?

minimum phase condition
holography
spectrum has only real zeros

* The significance of this is that real zeros are measured data points common to both the power and complex spectrum.

* *When does a bandlimited function have only real zeros?*

When are there sufficient real zeros present that their co-ordinates can be used to uniquely recover the complex function?

Conditions for 1D functions to have only real zeros

$$FT\{f(t)\} = F(x + iy)$$

i) f(t) positive, non-decreasing $t \geq 0$, and even (or odd)....F(x) real simple zeros

ii) f(t) positive, decreasing $t \geq 0$, and nowhere convex....F(x) only real zeros

iii) F(x) real then $F^n(x)$ only real zeros for some $n > N$

iv) $\alpha \leq f^1(t)/f(t) \leq \beta$ then zeros of F(z) lie in $\alpha \leq y \leq \beta$ for $f(t) > 0$

Polya:
"....could see no general rules..."

OTHER IDEAS.....

LOGAN'S THEOREM

F HAS NO ZEROS IN COMMON WITH
$\hat{F}$ AND BAND < OCTAVE ....THEN
REAL ZEROS CODE F
(BSTJ <u>56</u> '77 p487)

POLYA

$$F.T. \left\{ \left(1-|t|^{2n}\right)^{\alpha-1} \right\}$$ HAS REAL ZEROS
IF $\alpha \geq 0$, n INTEGER > 0

RODRIGUE'S FORMULA

$$\psi_n = \frac{1}{g(t)} \frac{d^n}{dt^n} \left[g(t) \left(1-t^2\right)^n \right]$$

HAS F.T. WITH REAL ZEROS.....

REAL ZEROS / ZERO CROSSINGS

... ALSO IMPORTANT IN ADAPTIVE OPTICS AND PHASE UNWRAPPING

... CONCEPT OF WAVEFRONT DISLOCATIONS OR "VORTICES"

ZERO SYMMETRIES

$f(t)$ REAL	$F(z)$ y AXIS SYM.
REAL, EVEN	x AND y
REAL, ≥ 0, CONCAVE AND MONOTONICALLY DECREASING	ON x AXIS

$$z_j \sim \frac{j\pi - i \ln\left[f(a)/f(-a)\right]}{2a}$$

Under what conditions do spectra have a preponderance of real zero crossings?

i.e. sufficient to provide a reconstruction of the image

Are these real zero crossings isolated points or lines or contours?

Is there anything in the literature?

1D or 2D bandlimited functions

Is there any insight from physical measurements?

Two dimensional bandlimited functions are generally irreducible.

The zero sheet of a polynomial with two complex variables is a 2-D surface embedded in a 4-D space.

Real zeros can only occur on:

(1) Closed contour
(2) Infinitely extended curve
(3) Isolated points

Suppose there are sufficient real zeros to represent complex function?

How do we know this?

Shannon sampling theorem and Hadamard product

How do we recover the function?

1D Hadamard product
2D Osgood product (?)

or estimation technique: local product representation

estimation/extrapolation method

EISENSTEIN'S CRITERION

(FIDDY, BRAMES & DAINTY OPT. LETT 8, 96 (1983))

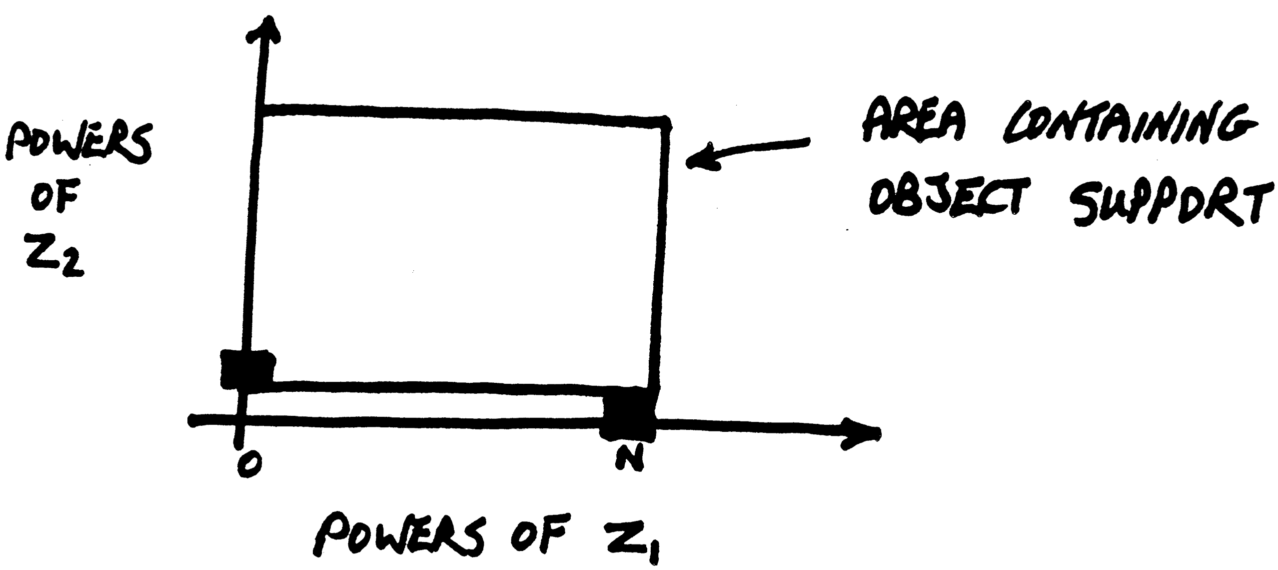
$F(z_1, z_2)$ IS IRREDUCIBLE IF THERE IS AN IRREDUCIBLE FACTOR $p(z_2)$ [$\equiv z_2$ FOR 2-D CASE] SUCH THAT FOR

$$F(z_1, z_2) = F_0(z_2) + F_1(z_2)z_1 + F_2(z_2)z_1^2 + \dots$$
$$\dots + F_N(z_2)z_1^N$$

$F_0, F_1, \dots F_{N-1}$ DIVISIBLE BY p

F_N NOT DIVISIBLE BY p

F_0 NOT DIVISIBLE BY p^2



lows:

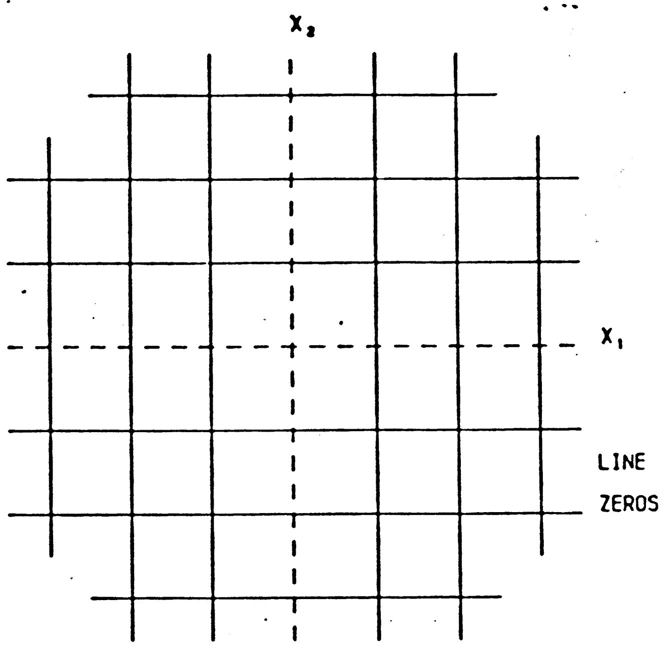


Fig. 1. The zero lines in mod(F) on the x_1–x_2 plane for a featureless square object.

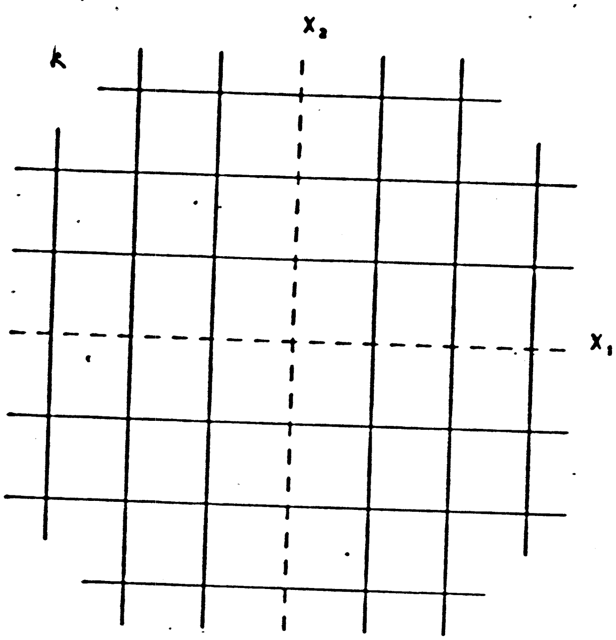


Fig. 3. The zeros of Re[$F(x_1, x_2)$], where f_{0M} is nonzero but small with respect to the rest of the object, which is the featureless square of Fig. 1.

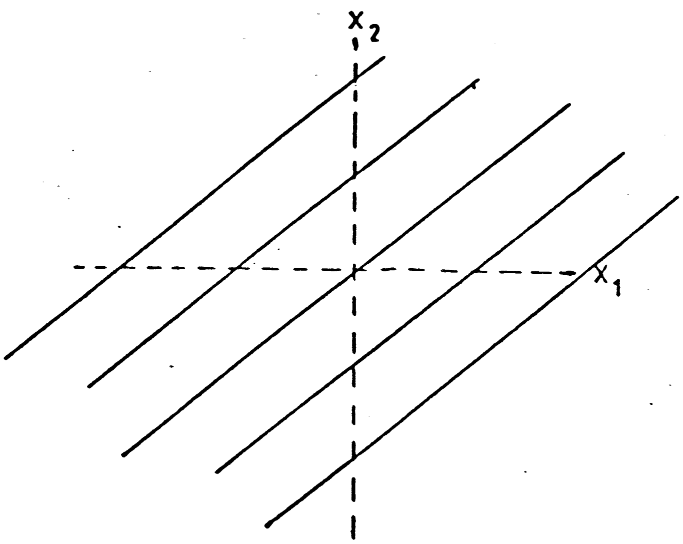


Fig. 4. The zeros of Im[$F(x_1, x_2)$], where f_{0M} is the same as for Fig. 3.

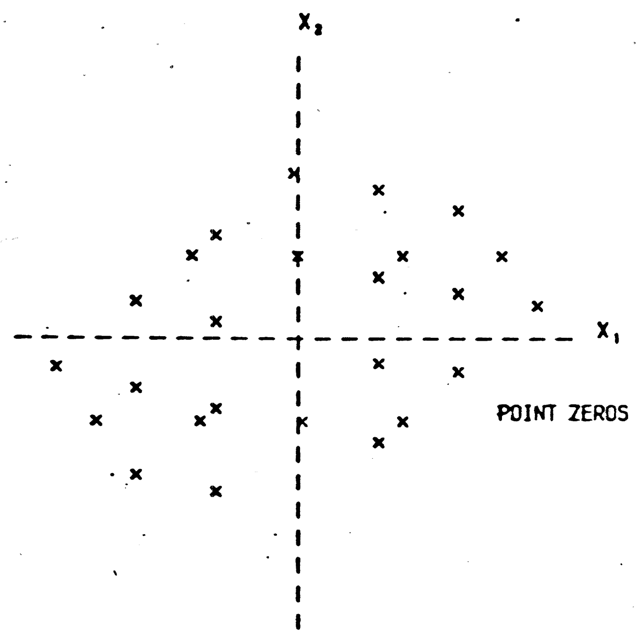


Fig. 5. The zeros of mod[$F(x_1, x_2)$], with f_{0M} chosen as for Figs. 3 and 4.

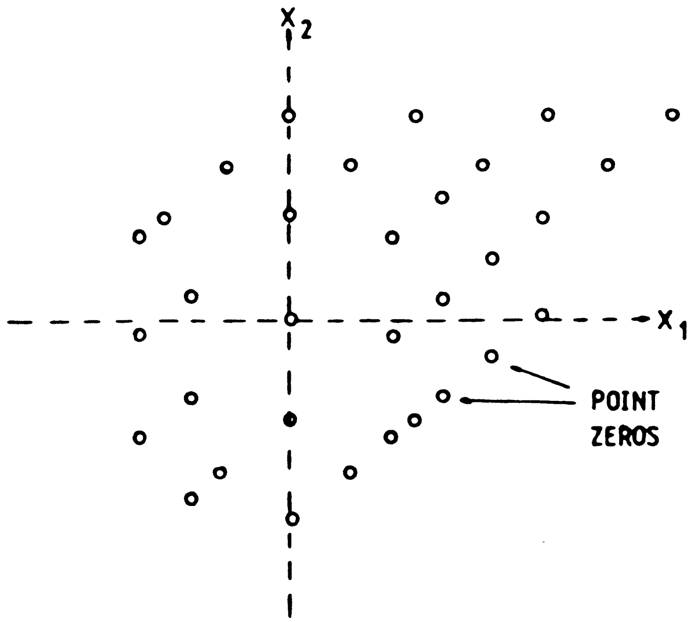
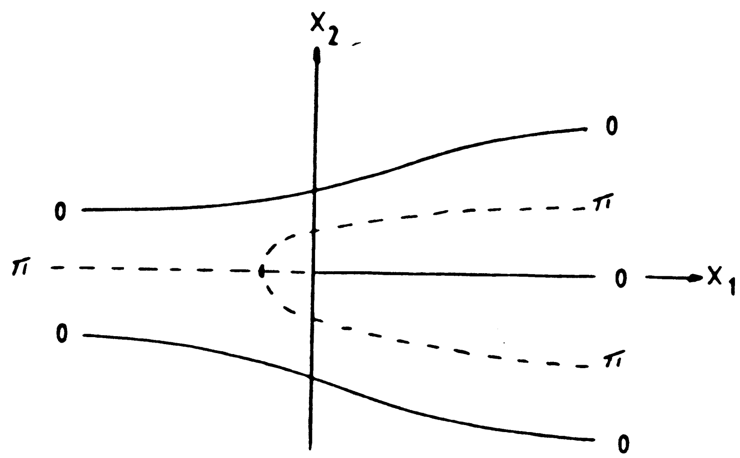
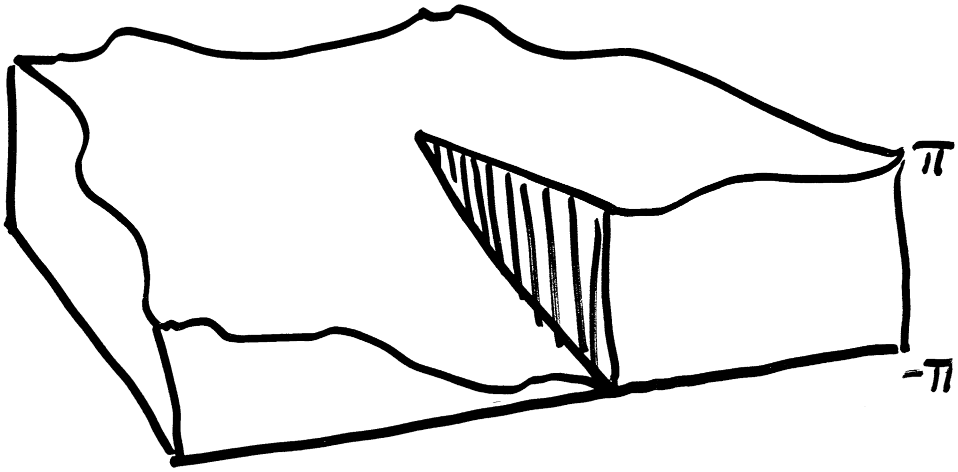


FIG: 6



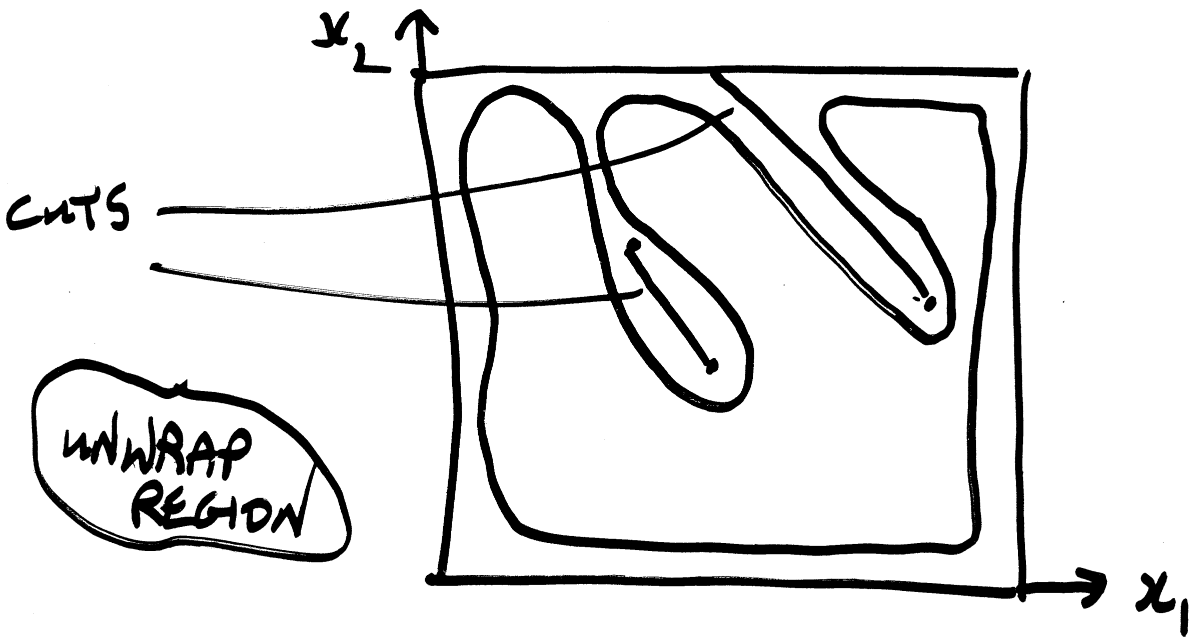
PHASE UNWRAPPING

2-D



— $F(x_1, x_2) = 0$ at points

— $\arg F(x_1, x_2)$ undefined at these points but analytic properties show a branch cut.



Solution from real zeros?

Product (Hadamard) formalism to get initial phase estimate, then iterate.

Chen, P-T, M. A. Fiddy, C-W. Liao and D. A. Pommet, "Blind deconvolution and phase retrieval using point zeros", J. Opt. Soc. of Amer. A, 13, pp1524-1531, 1996.

Use of spectral estimation technique (PDFT) from zero crossings

Liao C-W, M. A. Fiddy and C. L. Byrne, "Imaging of targets from intensity data", J. Opt. Soc. Amer. A 14, pp3155-3161, 1997.

Zero Phase Retrieval

Points at which 2D spectra of bandlimited functions are zero can be used to generate a polynomial approximation of the complex spectrum.

These point zero locations are common to both the spectrum and the associated power spectrum.

The phase of the complex valued polynomial along with the measured intensity data can be used to generate an initial guess for the function which an iterative (error-reduction) algorithm can improve.

The **Weierstrass preparation theorem** states that if a two-dimensional function $F(z_1,z_2)$ is analytic for $|z_1|< a$, $|z_2|< b$, let $F(0,0) = 0$, but $F(0,z_2) \neq 0$, then there is a polycylinder $|z_1|< \delta$, $|z_2|< r$ in which $F(z_1,z_2)$ can be represented as follows :

$$F(z_1,z_2) = g(z_1,z_2)(z_2^m + p_1(z_1)z_2^{m-1} + \cdots + p_m(z_1))$$

where g is analytic, $g(z_1,z_2) \neq 0$, $p_j(z_1)$ is analytic in z_1 for $|z_1|< \delta$ and $p_j(0) = 0$ (j = 1, ... , m).

$$z_2^m + p_1(z_1)z_2^{m-1} + \cdots + p_m(z_1) = 0$$

This equation is called a pseudoalgebraic equation, and this equation has m roots for each fixed z_1.

For each z_1, the function $F(z_1,z_2)$ can be approximated as a product form which is

$$F(z_1,z_2) \propto (z_2 - \eta_1)(z_2 - \eta_2)\cdots(z_2 - \eta_m)$$

where $\eta_1, \eta_2, \cdots, \eta_m$ (all of them depend on z_1) are zeros of $F(z_1,z_2)$.

Reconstruction From Point Zero Locations :

When the modulus of $F(u,v)$ goes to zero, the phase $Arg[F(u,v)]$ becomes indeterminate. In the neighborhood of a point zero associated with a complex factor, the field can be expanded as a Taylor series about this point

$$F(u,v) = (A_1 u + iA_2 v)$$

and the phase becomes

$$Arg[F(u,v)] = Arc\tan(A_2 v / A_1 u)$$

Provided $A_1 = 1$ and $A_2 = \pm 1$ (positive zero if $+1$, negative zero if -1), the phase will change by $\pm 2\pi$ for each circuit of a closed path around the point zero. This effect is also called a phase dislocation.

We first locate the real zero point locations from the Fourier intensity, then locally approximate $F(u,v)$ from the product of its zero point complex factors,

$$F(u,v) = (u - u_0) + iA_2(v - v_0),$$

for the point zero centered at (u_0, v_0).

If N real zero points are located, by setting the z_2 as the real plane, we can approximate the function $F(u,v)$ as the product of these N factors, i.e., by

$$F(u,v) = \prod_{n=1}^{N} (U_n + iA_n V_n),$$

where $U_n = (u - u_n)$, $V_n = (v - v_n)$, and A_n is the sign of nth zero.

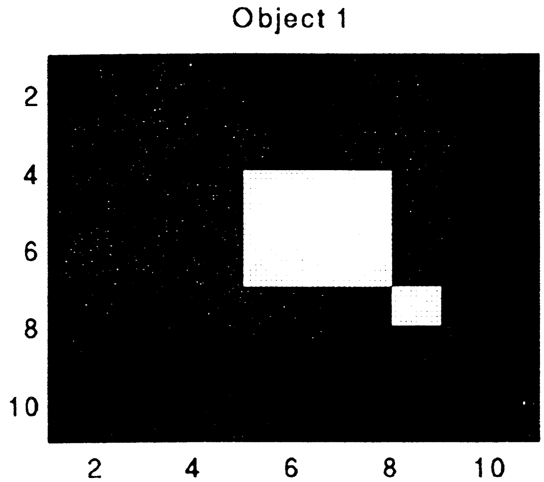


Figure 6.4

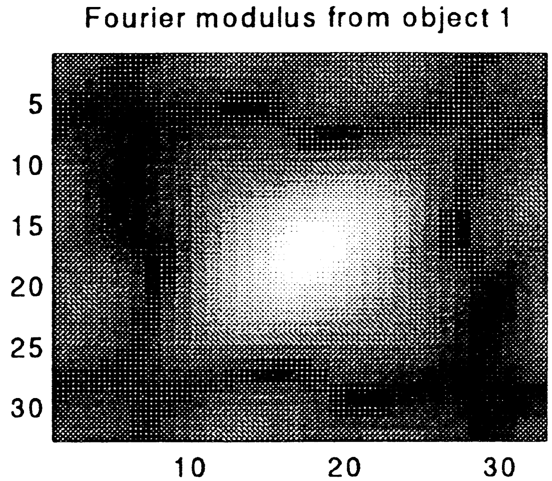


Figure 6.5

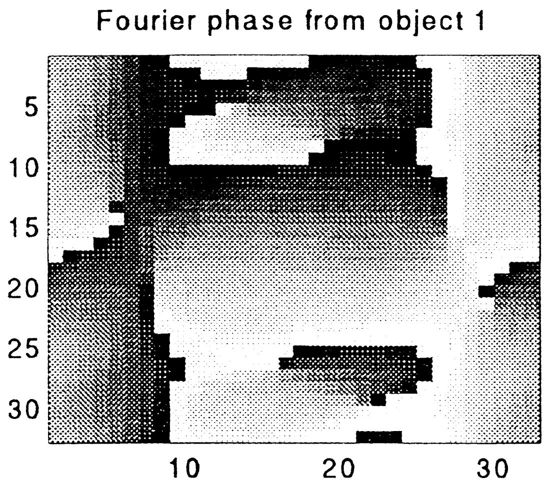


Figure 6.6

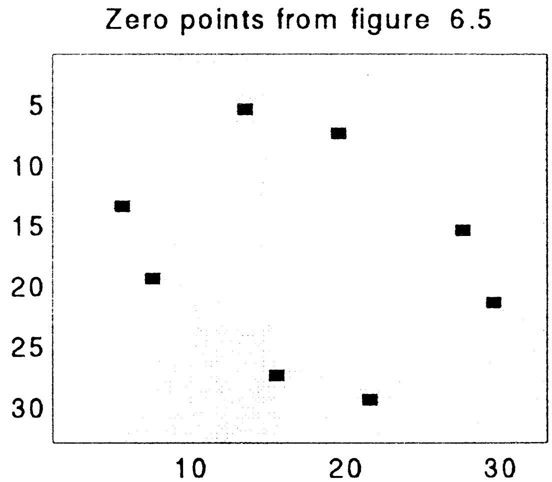


Figure 6.7

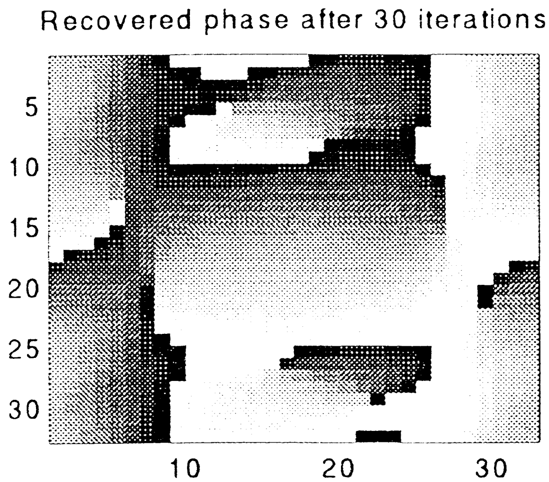


Figure 6.12

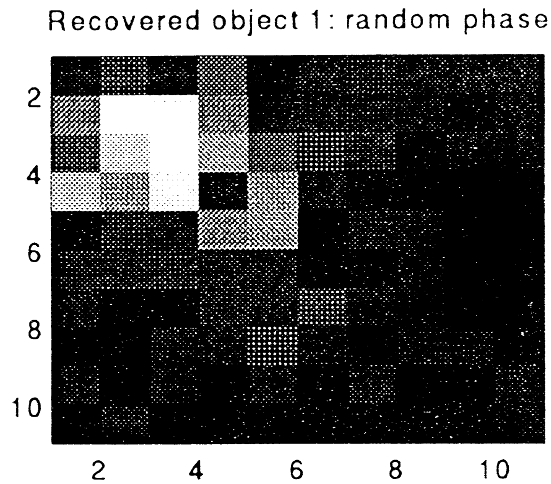


Figure 6.13

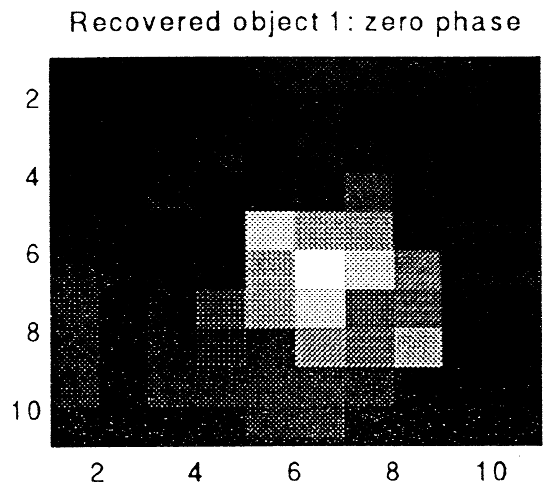


Figure 6.14

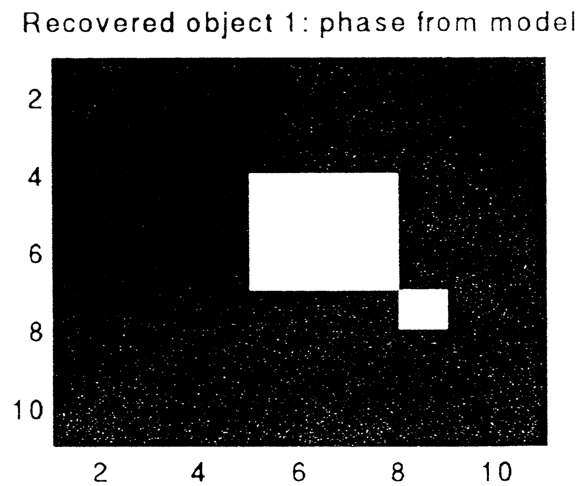


Figure 6.15

PDFT (Prior Discrete Fourier Transform)

- Use prior information to minimize error function:

$$\int_{-\pi}^{\pi} dr \frac{1}{P(r)} \left| V(r) - P(r) \sum_{n=-N}^{N} a_n e^{inr} \right|^2$$

- PDFT Relationships:

$$f(m) = \sum_{n=-N}^{N} a_n p(m-n) \quad ; \quad m = -N, \ldots, N$$

$$\mathrm{PDFT}(r) = P(r) \sum_{n=-N}^{N} a_n e^{inr}$$

PDFT AND PHASE RETRIEVAL

FIND ZEROS

ESTIMATE $\mathrm{Re}\{F\}=0$ CONTOURS

FROM THIS DEDUCE VALUES FOR $\mathrm{Im}\{F\}$ KNOWING $|F|^2$

ASSIGN SIGNS TO $\mathrm{Im}\{F\}$ CONSISTENT WITH SIGN CHANGE WHEREVER $|F|=0$.

USE THESE DATA TO ESTIMATE $\mathrm{Im}\{F\}$

USE $\mathrm{Re}\{F\}$ AND $\mathrm{Im}\{F\}$ ESTIMATES

PDFT for phase retrieval.

assume now that we have only the magnitude data $\{|F(m\Delta,n\Delta)|, |m|\leq M, |n|\leq N\}$, and Δ chosen so that π/Δ is greater than our estimate of s.

complex-valued function F(a,b) can be written as:

$$F(a,b) = R(a,b) + iQ(a,b),$$

where both R and Q are real-valued functions.

Letting r(x,y) and q(x,y) be the inverse Fourier transforms of R and Q, hence

$$r(x,y) = [\ f(x,y) + f(-x,-y)\ ]/2\ ,$$

and

$$q(x,y) = [\ f(x,y) - f(-x,-y)\ ]/2i\ .$$

Note: R(a,b) is real and symmetric, that is $R(-a,-b)=R(a,b)$.

Data values $|F(m\Delta,n\Delta)|$ that are (nearly) zero correspond to (nearly) zero values of both $R(m\Delta,n\Delta)$ and $Q(m\Delta,n\Delta)$.

We use (some of) these point zeros, along with our support information, to reconstruct $r(x,y)$ by means of the PDFT.

To reconstruct $r(x,y)$ from zeros of $R(a,b)$ via the PDFT we need at least one nonzero value of R; otherwise the PDFT estimate of r will be identically zero.

Since $F(-a,-b)=F(a,b)$, $F(0,0)$ is real; therefore $R(0,0)=|F(0,0)|$ or $R(0,0)=-|F(0,0)|$; we take $R(0,0)=|F(0,0)|$, arguing that $f(x,y)$ is often nonnegative in practice.

PDFT estimator of $r(x,y)$ is

$$r(x,y)'=\mathrm{PDFT}r(x,y)= p(x,y)\ \Sigma z\ c(m,n)\ \exp(i(m\Delta, n\Delta$$

where Σz denotes summation over (m,n) for which $(m\Delta,n\Delta)$ is one of the point zeros being used, as well as the pair $(0,0)$.

We find the coefficients $c(m,n)$ by forcing the Fourier transform to agree with $|F(0,0)|$ at the point $(0,0)$ and to be zero at the point zero locations; this is estimate $r(x,y)'$ of $r(x,y)$.

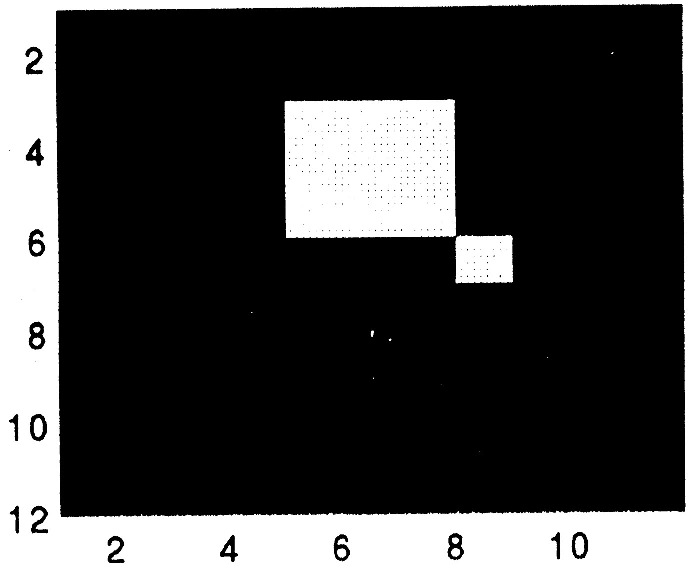


Figure 7-1a

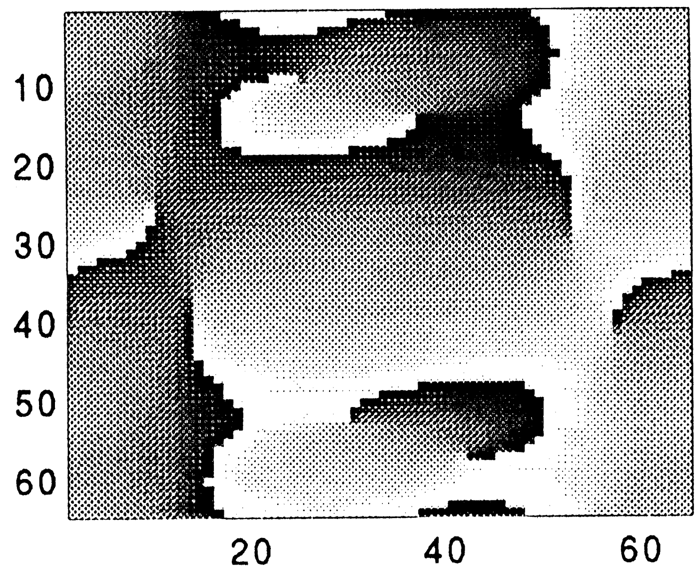


Figure 7-1b

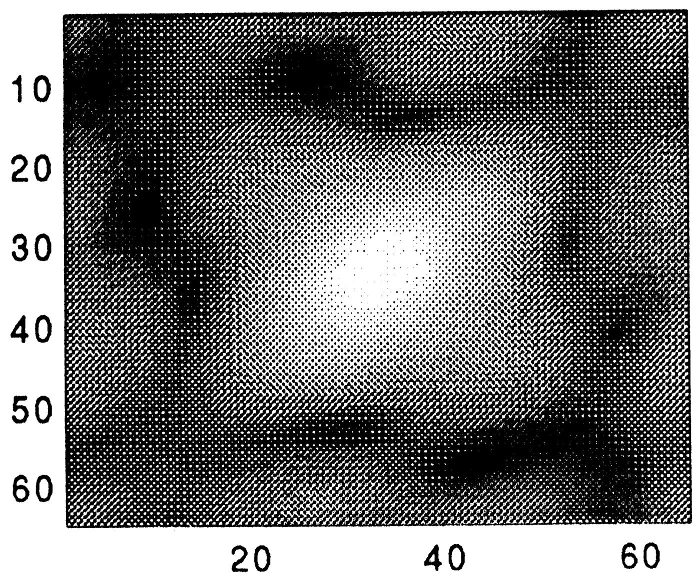


Figure 7-1c

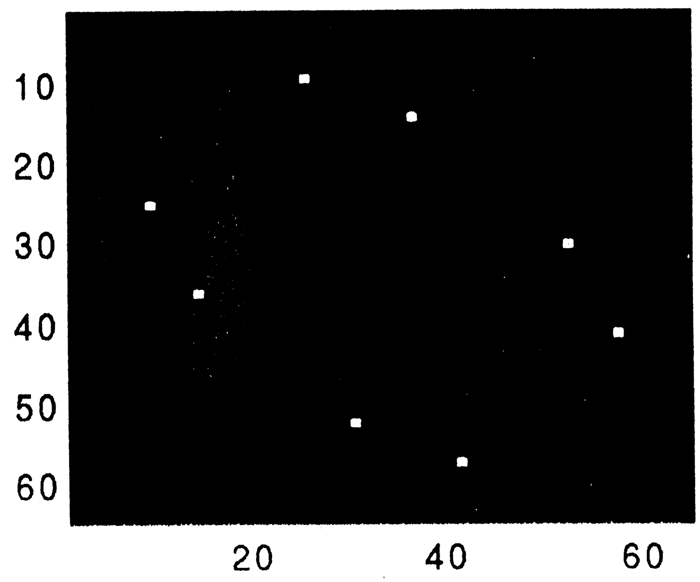


Figure 7-1d

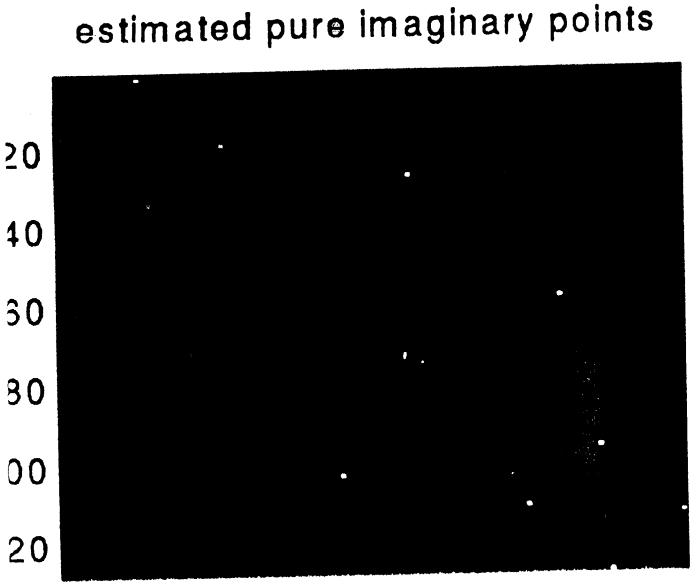


Figure 7-10a

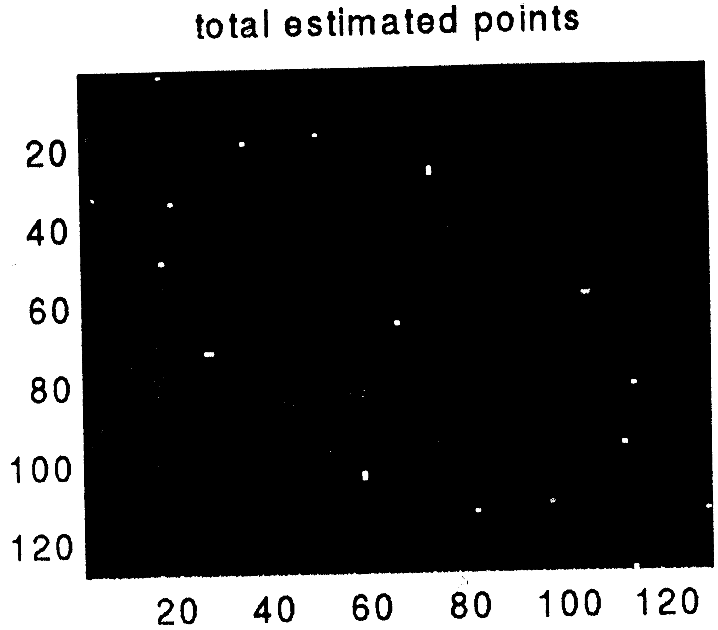


Figure 7-10b

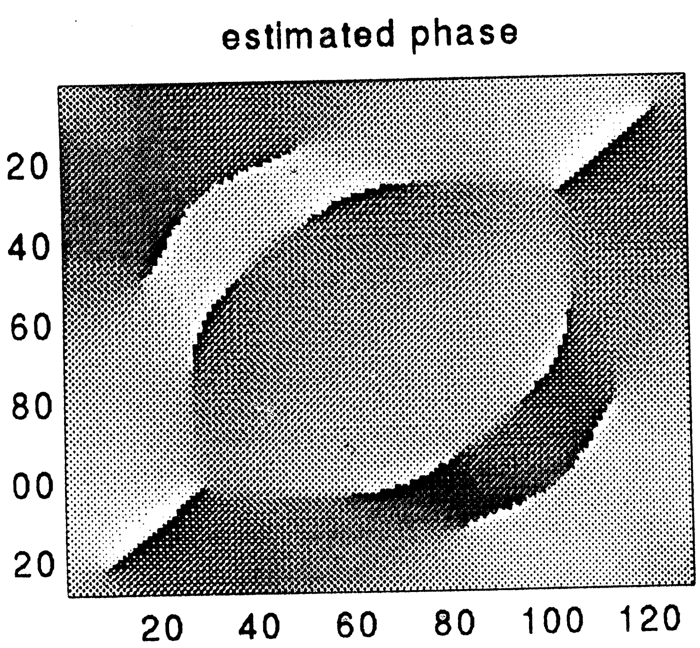


Figure 7-10c

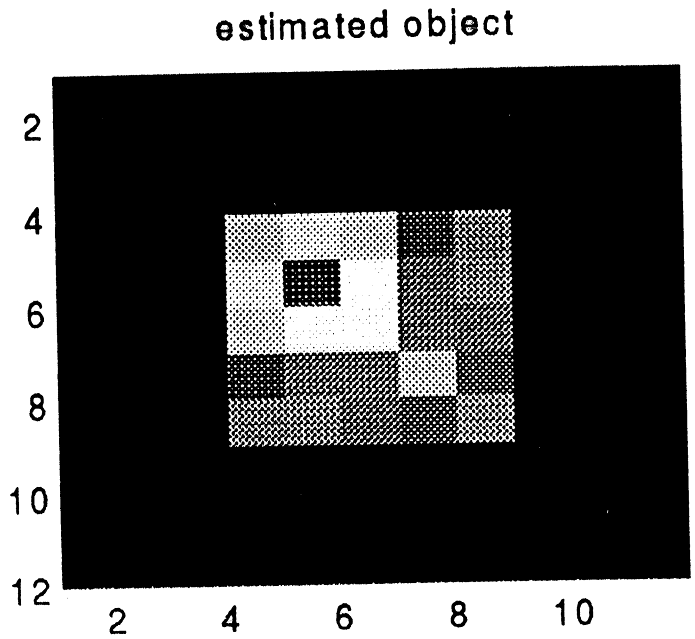


Figure 7-10d

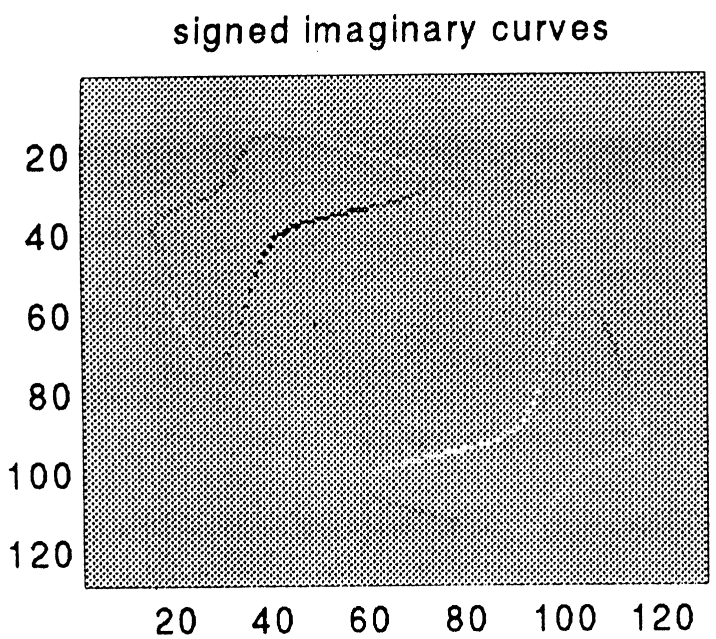


Figure 7-11a

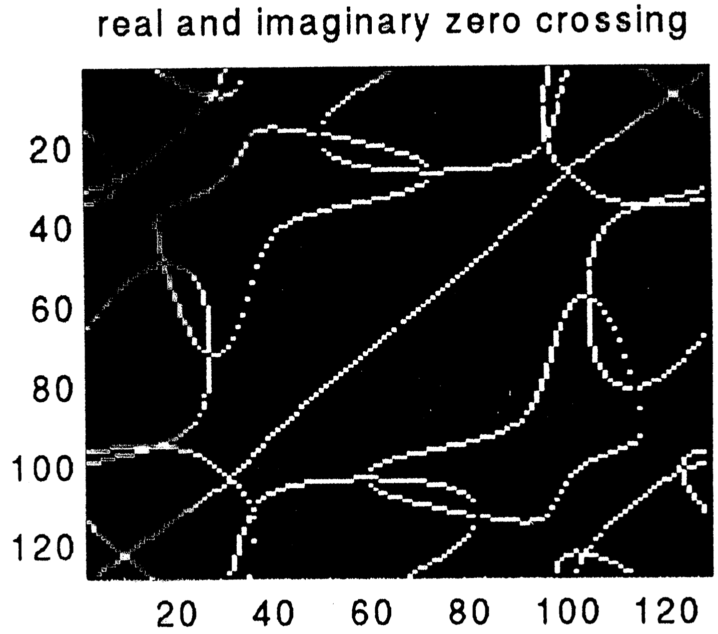


Figure 7-11b

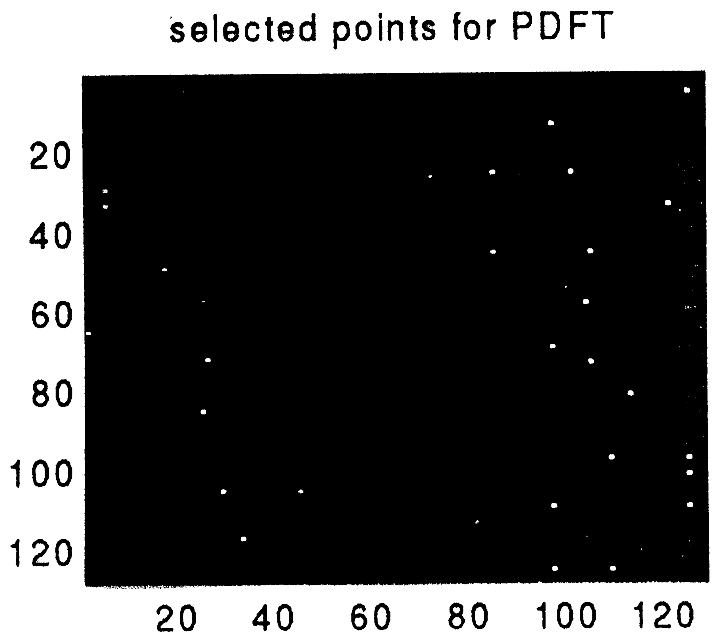


Figure 7-11c

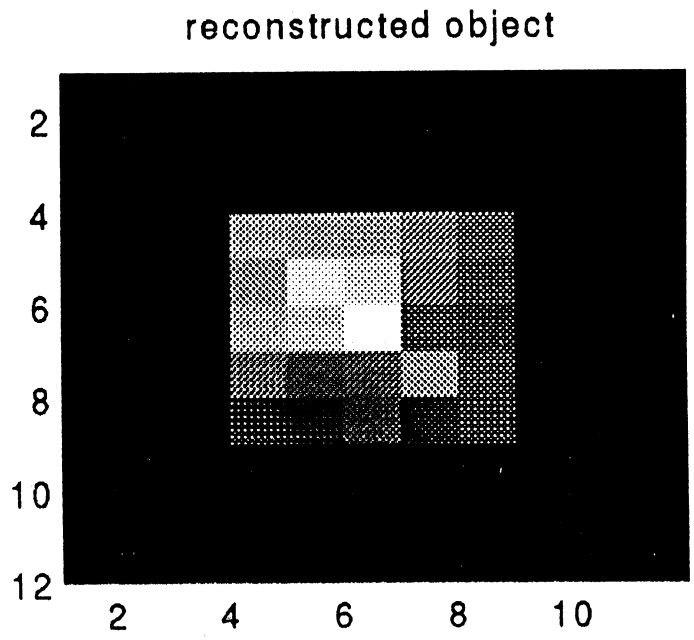
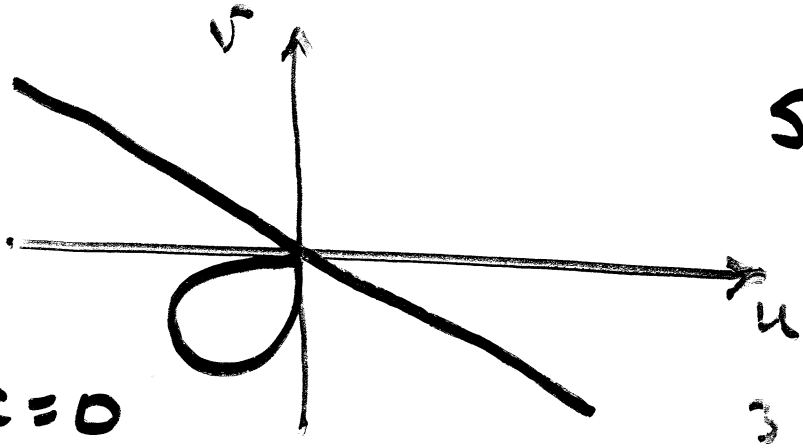


Figure 7-11d

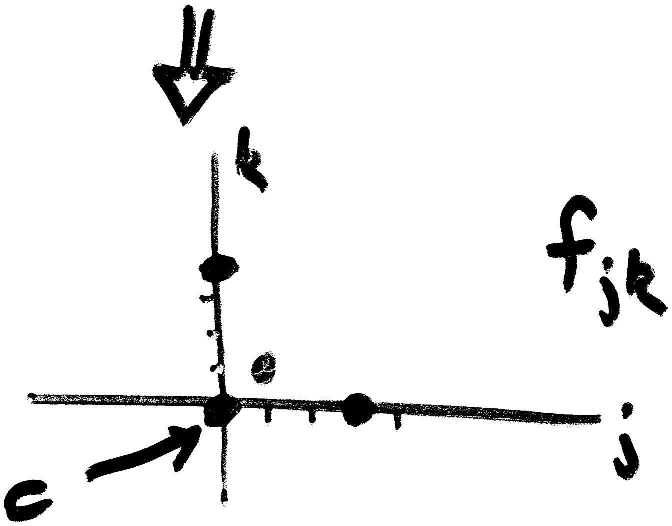
IRREDUCIBLE FUNCTION



$c = 0$

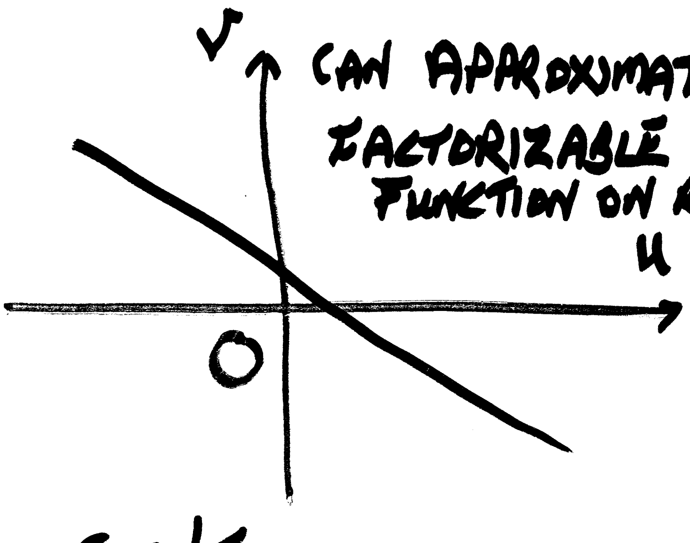
SATISFIES EISENSTEIN'S CRITERION

$u^3 + v^3 + uv = c$

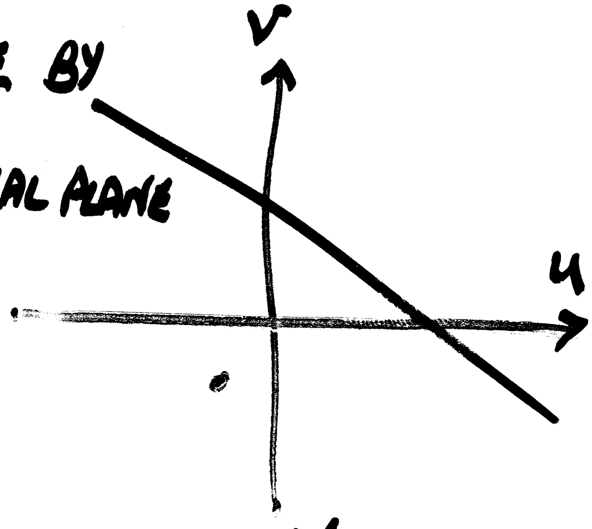


f_{jk}

CAN APPROXIMATE BY FACTORIZABLE FUNCTION ON REAL PLANE

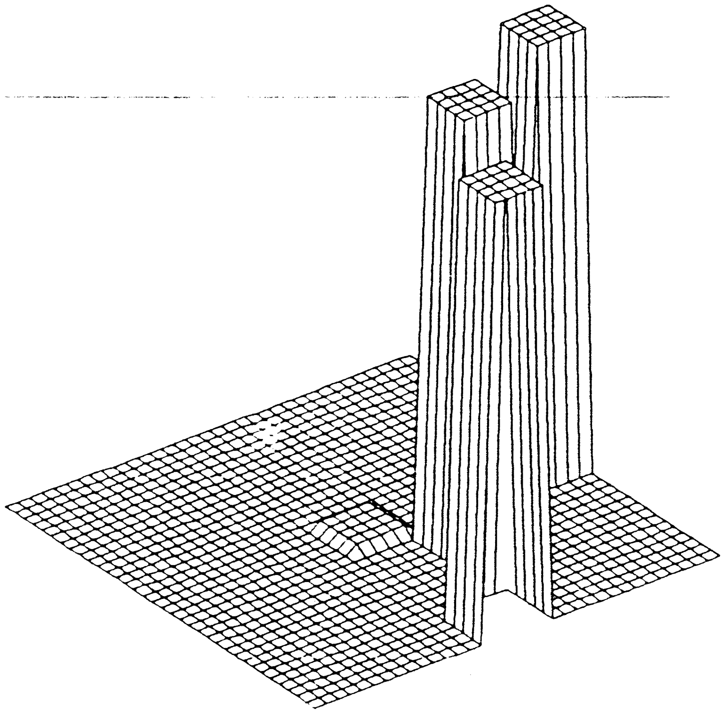


$c = 1/54$



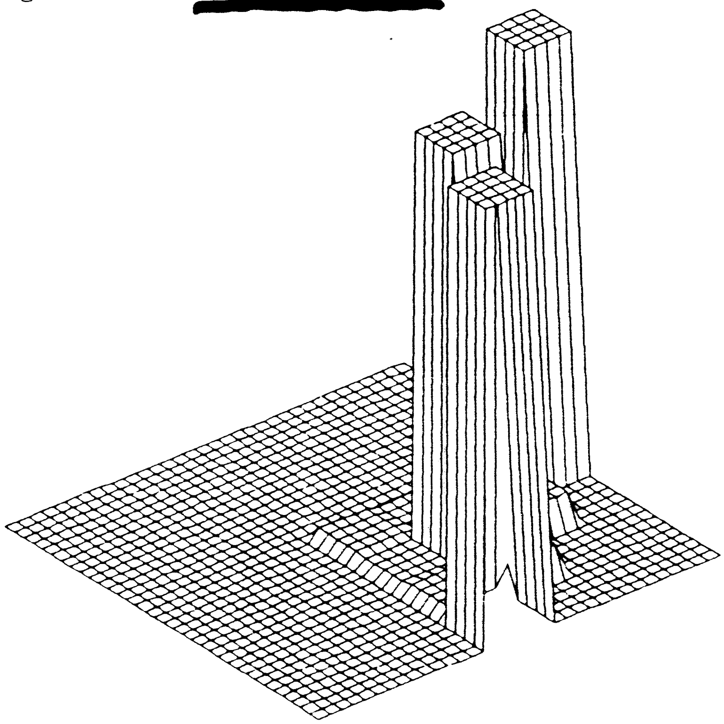
$c = 1/27$

original signal



signal from factored zero sheet without flipping

real plane



SIGNAL OR "IMAGE" ALMOST IDENTICAL TO ORIGINAL SIGNAL DESPITE REPRESENTING ITS SPECTRUM BY A FACTORIZABLE FUNKTION.

SUMMARY SO FAR

REAL ZEROS AVAILABLE FROM $|F|$

REAL ZEROS COULD ENCODE F

DIFFERENT APPROACHES TO USING REAL ZEROS TO CALCULATE F

STILL AN AMBIGUITY SINCE CAN APPROXIMATE IRREDUCIBLE F BY REDUCIBLE ONE. LOCAL FACTORIZATION MAY BE GOOD NEWS

ARE REAL ZERO LOCATIONS SUFFICIENT?

[redacted]

Significance of phase and amplitude in the Fourier domain

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Received March 5, 1997; revised manuscript received May 20, 1997; accepted May 22, 1997

We add new thoughts and aspects to the importance of phase and amplitude in the Fourier domain. We show how very similar objects react radically differently if, in the Fourier domain, either the phase was lost completely or the amplitude was modified to be constant. We also discuss the great influence of symmetry on the relative significance of the Fourier amplitude and of the Fourier phase. We show how changing the value of one pixel in some objects completely changes the significance of the Fourier phase and amplitude. © 1997 Optical Society of America [S0740-3232(97)00111-7]

1. DEFINITION OF THE PROBLEM

It is a widespread belief that the Fourier phase is more important than the Fourier amplitude. Even if one ignores the Fourier amplitude, it is still supposedly possible to recover a fairly good image from the Fourier phase alone. This belief was established several decades ago by Kozma and Kelly[1] and others. The synthetic-aperture radar community even named the radar signal they received "phase history," because they had only to preserve the phase accurately. One of the most famous papers regarding the importance of the Fourier phase was written by Oppenheim and Lim.[2] These authors used careful formulations when arguing about the validity of their findings, but the community largely adopted the plain view that the phase is more important than the amplitude.

Our aim is to shed some new light on this proposition. In Section 2 we will define quantitatively how one can answer the question of the relative significance of phase and amplitude. In Section 3 we will briefly review some cases that support the proposition. In Section 4 we mention an important counterexample. We do not have a complete answer to the question of which is more important, the Fourier phase or the Fourier amplitude. But we will present some numerical examples that indicate that the symmetry properties of the object have some effect. Also, it matters whether the object is real valued or somehow complex. A real-valued object causes the Fourier transform to be hermite symmetrical. Section 5 is dedicated to symmetrical objects and Section 6 is dedicated to asymmetrical objects. Some conclusions are drawn in Section 7.

2. QUANTITATIVE DESCRIPTION OF THE PROBLEM

The object, which may be complex, is defined as

$$u(x) = |u(x)|\exp[i\phi(x)]. \tag{1}$$

The associated spatial frequency spectrum is

$$\tilde{u}(\nu) = \int u(x)\exp(-i2\pi\nu x)\mathrm{d}x, \tag{2}$$

$$\tilde{u}(\nu) = A(\nu)\exp[i\alpha(\nu)]. \tag{3}$$

The question is, what is more important, $A(\nu)$ or $\alpha(\nu)$? One way to answer may be called test in isolation. For example, the phase factor $\exp[i\alpha(\nu)]$ is ignored. Hence, the remainder of the frequency spectrum is

$$A(\nu) = \tilde{v}_A(\nu). \tag{4}$$

The associated image is

$$v_A(x) = \int \tilde{v}_A(\nu)\exp(i2\pi\nu x)\mathrm{d}\nu. \tag{5}$$

The other kind of isolation yields

$$\exp[i\alpha(\nu)]\mathrm{rect}\left(\frac{\nu}{\Delta\nu}\right) = \tilde{v}_F(\nu). \tag{6}$$

The rect function confines $\tilde{v}_F(\nu)$ into the bandwidth of the Fourier amplitude $A(\nu)$. The phase-only output is

$$v_F(x) = \int \tilde{v}_F(\nu)\exp(i2\pi\nu x)\mathrm{d}\nu. \tag{7}$$

In many cases the image qualities of both $v_A(x)$ and $v_F(x)$ will be obvious, either quite good or quite poor.

To make a more general statement, one might compute ensemble averages of normalized correlations:

$$C_A = \frac{\int v_A(x)u^*(x)\mathrm{d}x}{\left[\int |v_A(x)|^2\mathrm{d}x\right]^{1/2}\left[\int |u(x)|^2\mathrm{d}x\right]^{1/2}}, \tag{8}$$

$$C_F = \frac{\int v_F(x)u^*(x)\mathrm{d}x}{\left[\int |v_F(x)|^2\mathrm{d}x\right]^{1/2}\left[\int |u(x)|^2\mathrm{d}x\right]^{1/2}}. \tag{9}$$

One can pursue this correlation approach easily by invoking the Parseval theorem, which converts those integrals into the Fourier domain:

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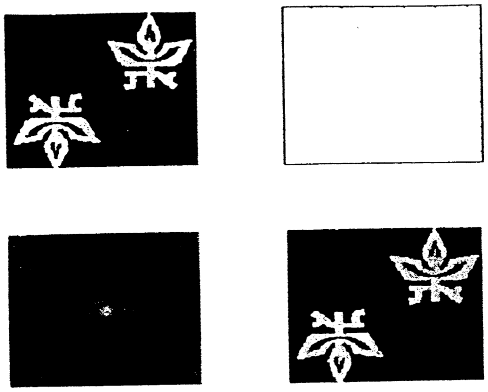

Fig. 7. Reconstructions from an asymmetric object with zero phase: peak phase zero amplitude-only reconstruction (upper left), peak phase zero phase-only reconstruction (upper right), peak phase 90° amplitude-only reconstruction (lower left), peak phase 90° phase-only reconstruction (lower right).

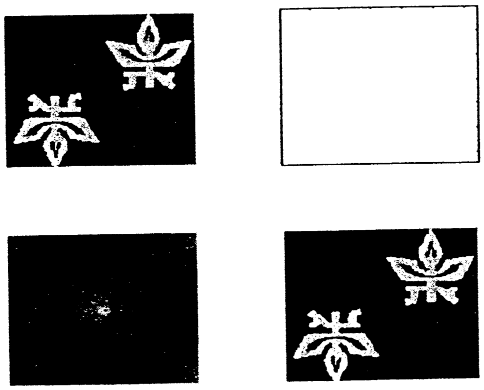

Fig. 8. Reconstructions from an asymmetric object with totally random phase: peak phase zero amplitude-only reconstruction (upper left), peak phase zero phase-only reconstruction (upper right), peak phase 90° amplitude-only reconstruction (lower left), peak phase 90° phase-only reconstruction (lower right).

- Peak phase zero or 90°.
- $u(x - x_0)$ phase constant or random.

The outputs are shown in Figs. 7 and 8. The most remarkable result here is the generation of a twin image in Figs. 7 and 8. Again, the phase shift of the central peak has a major influence on the phase-only and amplitude-only outputs.

7. CONCLUSIONS

Computer experiments showed that imposing a change on one pixel of an object may switch the importance of the Fourier phase and the Fourier amplitude. We examined the influence of symmetry and asymmetry of objects on the significance of phase and amplitude in the Fourier domain. It is our conclusion that there is no strict confirmation of the proposition "phase is more important than amplitude." Several object properties such as symmetry and reality play an important role in determining the significance of phase and amplitude in the Fourier domain.

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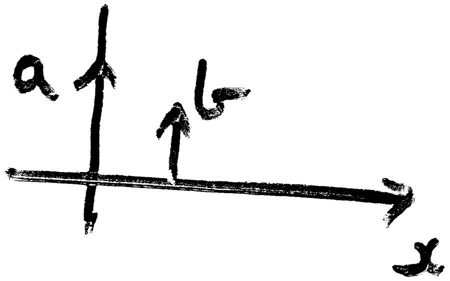
WHAT REALLY DISTINGUISHES THESE TWO SITUATIONS?



$$\Rightarrow F.T.\{a\delta + b\delta(x-1)\}$$

$$= A + Be^{iu}$$

$$= (A + B\cos u) + i(B\sin u)$$

IF $A \simeq B$, ZEROS OCCUR WHEN

$1 + \cos u \simeq 0$ <u>AND</u> $\sin u \simeq 0$

WHEN $u = \pi, 3\pi, \ldots$

PHASE: $\tan^{-1}\left(\dfrac{\sin u}{1+\cos u}\right)$ ROUCHE: $\simeq$ MIN PHASE?

.... MAGNITUDE ENCODES ALL INFORMATION....

PHASE COULD BE $-\pi/2 \leq \varphi \leq \pi/2$

$$A \rightarrow iA = Ae^{i\pi/2}$$

SPECTRUM IS $iA + Be^{iu}$

REAL POINT ZEROS WHEN

$$B\cos u = 0 \quad \text{AND} \quad A + B\sin u = 0$$

$\therefore$ ZEROS WHEN $u = \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$
IF $A \simeq B$.

PHASE: $\tan^{-1}\left(\frac{1+\sin u}{\cos u}\right)$ <u>NOT</u> MIN PHASE

MAGNITUDE HAS REAL ZEROS <u>BUT</u> PHASE VARIES DRAMATICALLY

WAVEFRONT DISLOCATIONS / VORTICES ASSOCIATED WITH REAL ZEROS.

CONCLUSIONS

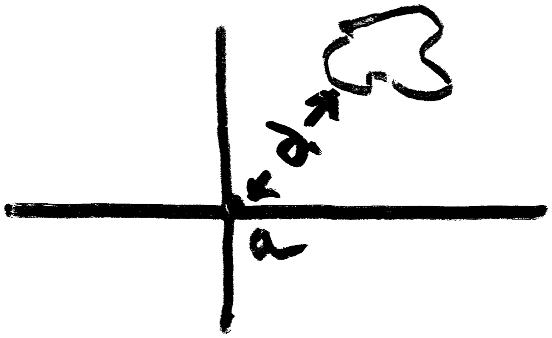
ZEROS CODE F

REAL ZEROS MAY CODE F IN MOST CASES

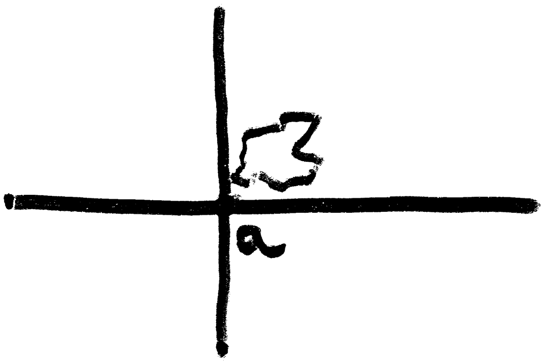
IF 'MINIMUM' PHASE, $|F|^2$ CODES F DIRECTLY

IF NOT 'MINIMUM' PHASE NEED TO ESTIMATE F FROM REAL POINT ZERO LOCATIONS.

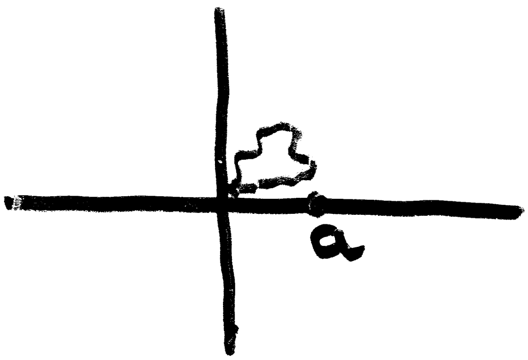
'REFERENCE' POINTS IN OBJECT DOMAIN APPEAR TO BE IMPORTANT



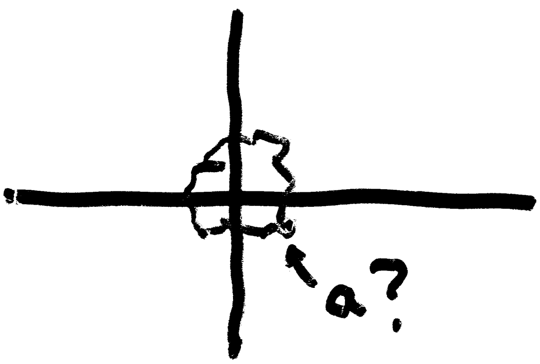
a 'small'
d 'large'
OFF AXIS HOLOGRAM



a 'large'
ROUCHE /MIN PHASE



a 'small'
EISENSTEIN/IRREDUCIBLE



a ?
GENERAL CASE /IRREDUCIBL

CONCLUSIONS CONTINUED

REFERENCE POINTS GENERALLY PROVIDE MECHANISM FOR PHASE RETRIEVAL OR JUSTIFICATION OF UNIQUENESS.

REAL ZERO CROSSINGS OF $|F|^2$ ARE ALSO AT LOCATIONS OF ZEROS OF F.

THE SET OF FUNCTIONS HAVING A SPECIFIC DISTRIBUTION OF REAL (POINT) ZEROS IS CONVEX.

EITHER ITERATIVELY OR BY THE DIRECT APPROACHES DESCRIBED, THE REAL ZEROS HOLD THE KEY TO EFFICIENT PHASE RETRIEVAL.

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